Two-Stage Game Theoretic Modeling of Airline Frequency and Fare Competition

Reed Harder, Vikrant Vaze

Abstract

Airlines make decisions about pricing and daily service frequency in a competitive environment. We develop a two-stage game-theoretic model of airline competition, where airlines make frequency decisions during the first stage and fare decisions during the second stage with knowledge of the first stage decisions. We prove that for a simplified two-player form of this game, with assumptions of unrestricted seats-per-flight and only non-nonstop passengers, the first-stage payoff function of each player is concave with respect to that player's frequency strategy. With the same assumptions, we also prove that this two-stage game belongs to the class of sub-modular games. Concavity and sub-modularity are shown by numerical experiments to hold for one player, two player, and three player games across a wide range of parameter values, with quadratic functions of player frequencies providing a good approximation ($R^2 > 0.9$) for airline payoffs in all cases. We use solve this model for an 11-airport, four-airline network using the myopic best-response learning heuristic, and the frequency predictions from this solution are validated against actual frequency data from this network. This paper demonstrates that a two-stage frequency-fare game of airline competition can exhibit properties (concavity and sub-modularity) that allow for a computationally tractable equilibrium solution across a wide range of parameter values and a good fit with observed airline frequencies.

Introduction

Airlines make fare and capacity allocation decisions in a competitive environment. Capacity decisions, encompassing decisions about seats-per-flight and frequency of service, affect both the operating costs and fare revenues of airlines. These decisions have significant implications for the performance of the air transportation system as a whole: over- and under-allocation of airline capacity has been shown to result in billions of dollars in costs to airlines and passengers (Ball et al., 2010; Kahn, 1993), wastage of system resources (Vaze and Barnhart, 2012a; Morisset and Odoni, 2011), passenger inconvenience (Barnhart et al., 2014; Wittman, 2014) and environmental damages (Schumer and Maloney, 2008). Decisions about capacity and fare are typically made sequentially, on different timelines. Capacity decisions are often made weeks or months in advance of the flights in question, with only an approximate understanding of future fare decisions. On the other hand, fare decisions can be made days or even minutes in advance with knowledge of prior scheduling. Moreover, frequency and fare decisions of different airlines are interdependent, both serving as tools in an airline's competitive arsenal: one could expect a higher frequency or lower fare than competing airlines to be more attractive to passengers seeking to travel on a particular route. The effects of frequency and fare decisions on passenger demand and the interdependency of these decisions between airlines competing across a network of airports means that they are important factors in the performance of the airline transportation system as a whole, and suggests the importance of developing tractable models that accurately describe their dynamics. Vaze and Barnhart (2012a, 2012b), for example, studied the role of airline frequency competition in airport congestion, and used a model of airline frequency competition to develop a strategy for reduction of passenger and flight delays as well as an improvement in airline profits.

In this paper, we develop a two-stage game theoretic model of airline competition, demonstrate its tractability across a range of assumptions and parameter values, and validate its predictions against observed airline behavior. This two-stage game approach accounts for both the interdependence of competing airline behaviors and the sequential nature of frequency and fare decisions used in practice. Furthermore, daily frequencies of airlines within a market tend to hold roughly constant for multi-month time periods (see **Figure 1**), suggesting that some form of frequency equilibrium may be a solution concept worth exploring. We assume that seats-per-flight are held constant and that all passengers are nonstop. We first prove analytically that for a simplified version of this model, with two airlines competing in a market with infinite seating capacity and the absence of a no-fly option for passengers, that the payoff function of each airline in the first stage frequency game is concave with respect to that player's frequency strategy. Additionally, we prove analytically that this simplified model belongs to the class of *sub-modular* games, and that for two-player games, changing the sign of the strategy space transforms the game into a *super-modular game*. We then extend our model, relaxing our assumptions of unlimited seating and the absence of a no-fly option for passengers, and demonstrate numerically that for a range of passenger utility and aircraft capacity parameter values, concavity and sub-modularity properties hold by fitting airline payoffs to quadratic functions of airline frequencies with high (> 0.9) values of R^2. These concavity and sub-modularity results allow us to formulate a suitable equilibrium solution concept and employ a tractable solution heuristic. We use this heuristic to solve for subgame perfect Nash Equilibrium for four airlines making frequency decisions across a network of 11 airports in the western United States, using supply and demand data from 2007 available on the Bureau of Transportation Services (BTS) online database. The frequency estimates from this solution are then validated against the observed frequencies of these airlines over the same period, which we use to calibrate the payoff functions of each airline to find a reasonable concordance of predicted and observed frequencies.

A number of previous studies have taken a game-theoretic approach to modeling frequency and fare competition. Hansen (1990) solved an airline frequency competition game using successive airline profit optimizations for 52 U.S. airports and 28 airlines, though model predictions showed some significant divergences with empirical data. Adler and colleagues published a series of studies using simultaneous equilibrium approaches to model frequency, seat allocation and fare decisions (Adler, 2001; 2005) or fare decisions alone (Adler and Smilowitz, 2007) as the second stage of a two-stage game (following route planning in the first stage), and simultaneous frequency, seat allocation and fare decisions as a one stage game (Adler, Pels, and Nash, 2010). Other studies solve single-stage game theoretic models to Nash equilibrium for frequency decisions (Vaze and Barnhart, 2012a; 2012b; and 2015), fare decisions (Aguirregabiria and Ho, 2012), simultaneous frequency and fare decisions (Hong and Harker, 1992; Pels, Nijkamp, and Rietveld, 2000; Zito, Salvo and La Franca, 2011; and Hansen and Liu, 2015), simultaneous frequency and seat allocation decisions (Wei and Hansen, 2007), and simultaneous frequency, fare and seat allocation decisions (Brueckner, 2010).

Treatment of two-stage frequency-fare games is more limited, and focused on two airline competition in a single market. Dobson and Lederer (1993) use heuristic methods to solve a single-market, two-airline game, with airline schedules decided before fares. Schipper, Rietveld, and Nijkamp (2003) analyze the shift from monopoly to duopoly equilibria following airline deregulation by simulating a single-market, two-airline two-stage frequency-fare game. Brueckner and Flores-Fillol (2007) analytically compare the properties of single-market, two-airline games with simultaneous frequency and fare decisions and those with sequential frequency and fare decisions. Hansen and Liu (2015) present a numerical example of single-market, two-airline two-stage game. Several studies have stressed the need to develop two-stage capacity fare game theoretic models in order to account for the sequential nature of these decisions (Dobson and Lederer, 1993; Norman and Strandenes, 1994; Schipper, Rietveld and Nijkamp, 2003; Brueckner and Flores-Fillol, 2007; Hansen and Liu, 2015), but there are very few analytical, computational, or empirical results available for such models.

This research contributes to the airline competition literature by demonstrating the tractability of a two-stage frequency-fare competition game that accurately reflects airlines' sequential decision-making process, and efficiently solving this model and validating the results against observed airline behavior in an 11-airport network with four competing carriers. It is the first study to prove concavity and sub-modularity properties of such a game analytically for a simplified case, and to numerically demonstrate the robustness of these properties in more complex models. These properties allow us to ensure a tractable framework that can be used to apply our two-stage frequency-fare model to more realistic competitive networks than the single-market, two-airline case typically considered for this class of games.

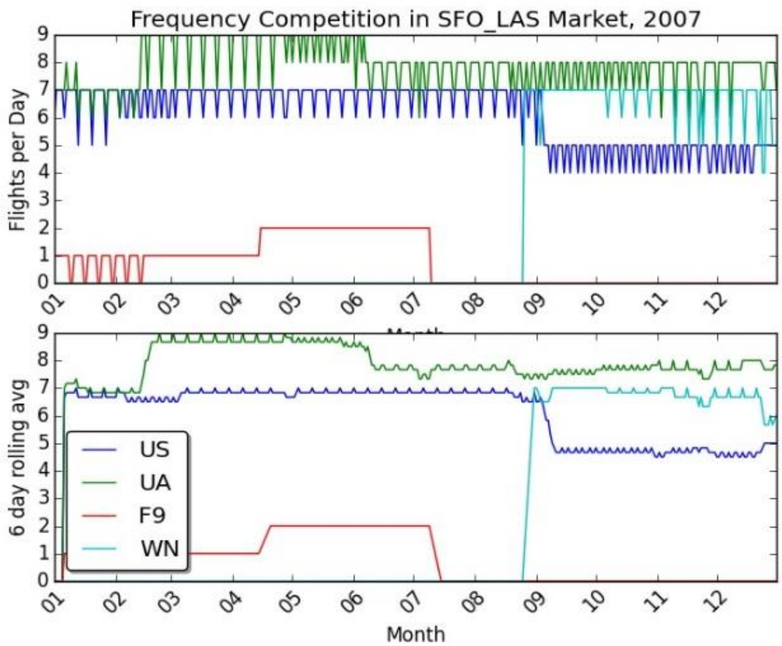


Figure 1: Flights per day, daily (top) and six day rolling average (bottom) of carriers competing in the SFO-LAS market, over the course of 2007.

Model

The two-stage game theoretic framework we use in this paper models frequency decisions as the first stage of each airline over a network, and the second stage as the fare pricing decisions of those airlines. While airline capacity allocation decisions modeled in the first stage of our game also include decisions about number of seats per flight in practice, we focus on frequency decisions as the critical component of capacity-based competition. While frequency and seats-per-flight decisions both affect airline carrying capacity, frequency decisions also significantly affect the attractiveness of a particular airline to passengers, with higher frequencies providing passengers with more flexible and convenient travel schedules (Belobaba, P. 2009a). In addition, major airlines typically serve short and medium haul U.S. markets using a fleet with low variability in the number of seats per flight, while frequency decisions in such markets often varies significantly both across years on the same origin-destination segment, and between segments in the same year. **Table 1** displays the average values of the coefficient of variation (the ratio of mean to standard deviation) of frequency and seats-per-flight across years and across operating segments for major operating segments, with values calculated using data from the Bureau of Transportation Statistics Air Carrier Statistics Database (BTS, 2015a).

Airline	Across Years		Across Segments	
	Seats-per-Flight	Frequency	Seats-per-Flight	Frequency
American Airlines (AA)	4.7%	12.7%	13.3%	60.2%
Delta Airlines (DL)	6.3%	21.7%	15.9%	67.9%
United Airlines (UA)	5.3%	27.3%	21.7%	67.1%
US Airways (US)	6.3%	16.9%	15.4%	58.6%
Southwest Airlines (WN)	3.5%	16.6%	1.8%	70.0%

Table 1: Variability in Seats-per-Flight and Frequency: Measured as Average Values of Coefficients of Variation

In this paper, frequency decisions are estimated as the equilibrium found by solving the interdependent payoff maximization problems for each airline. We define a market as an origin-destination pair of airports, and notate the set of markets being competed in as *R.* We denote the set of competing airlines as *A.*

The payoff function for each airline is given as the difference between the revenue and cost of operation for each market. Revenue for an airline *a* in market *m* is computed as

$$Rev_{a,m} = \min(M_m(MS_{a,m}), f_{a,m}s_{a,m})p_{a,m}$$

Where $M_{a,m}$ is the size of the market for that origin-destination pair, $s_{a,m}$ is the seating capacity per flight, and $MS_{a,m}$ is the airline market share. We model market share using a multinomial logit model, an approach widely used in prior literature for air travel demand (Garrow, 2012). We explore two random utility formulations of airline market share, one based on the often-cited "s-curve" relationship between frequency share and market share, and the other based on the concept of schedule delay, or the average difference between departure times and departure times desired by passengers. In the former, logit passenger utility is given by a linear combination of fare and a logarithmic transformation of frequency, consistent with the S-curve model of the relationship between market-share and frequency-share (Hansen and Liu, 2015; Vaze and Barnhart, 2015). Thus, for airline *a* and market *m* and the set of airlines competing in *m* notated as A_m, with the positive parameters α and β (coefficients of utility of frequency and fare, respectively), $p_{a,m}$ as the fare, $f_{a,m}$ as the frequency, and exponential of the utility of the no-fly option N_m, the market share of carrier *a* can be expressed as

$$MS_{a,m} = \frac{e^{\alpha\, ln(f_{a,m}) - \beta p_{a,m}}}{N_m + \sum_{i \in A_m} e^{\alpha\, ln(f_{i,m}) - \beta p_{i,m}}} \quad (1)$$

With prices equal and in the absence of a no-fly alternative, frequency share alone determines market share, frequency share in (1) completely determines market share, following an s-shaped curved whose shape is modulated by α. Market share can also be captured using the schedule-delay model, as discussed in (Hansen and Liu, 2015). In this case, market share for airline *a* takes the following form:

$$MS_{a,m} = \frac{e^{-\varphi {f_{a,m}}^{-r} - \beta p_{a,m}}}{N_m + \sum_{i \in A_m} e^{-\varphi {f_{i,m}}^{-r} - \beta p_{i,m}}} \quad (2)$$

Here φ and *r* are positive parameters modulating the utility of frequency. Hansen and Liu (2015) argue that this model describes a more plausible relationship between frequency share and market share. For instance, market share in this model depends on both frequency share and competitor frequency, such that an airline cannot simply dominate the market share of an already high frequency market by arbitrarily increasing its own frequency, as in the s-curve formulation.

Airline operating cost is typically modeled as a linear function of frequency in airline competition literature. An exception to this is the use of the Cobb-Douglas function (Adler and Berechman, 2001; Adler, 2001; 2005), expressed as $[\sum_{k\in Arc}(f_k)^{\alpha}]^{\beta}$, with Arc being the set of all legs in an airline's network, and f_k being the number of flights per week offered on leg k, for parameters α, β both > 0, estimated as $\alpha = 1.2$ and $\beta = 0.7$ by Adler (2005). With these values, we found this function to be approximated by a linear function with an R^2 of 0.9978 over a plausible range of frequencies (see **Figure 1**).

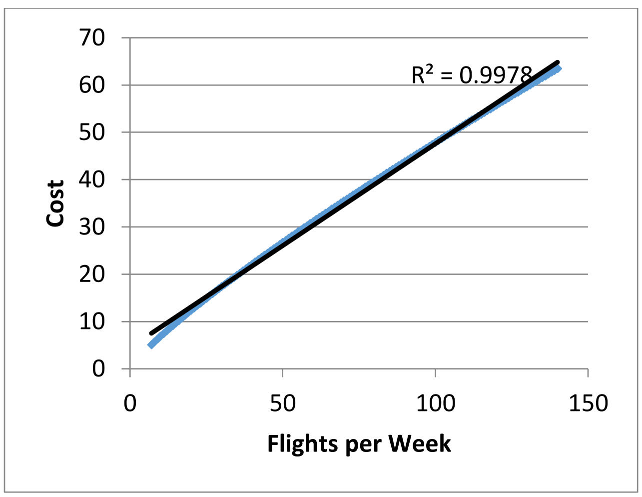


Figure 1: Cobb-Douglas cost function, fitted to a linear function of frequency

Thus, with $Cost(a,m) = c_{a,m}f_{a,m}$ for airline a and market m, where with $c_{a,m}$ as cost per flight in market m, the payoff function of airline a over a network of airports is given by

$$\pi_a = \sum_{m\in M_a} \min(M_{a,m}MS_{a,m},\ f_{a,m}s_{a,m})\, p_{a,m} - c_{a,m}f_{a,m} \quad (3)$$

Analytical Results for a Simplified Model

We now turn to a simplified version of the above model to analytically prove the following theorems for either market share function, (See Appendix I for full proofs), assuming the above payoff function in the single market, two carrier case, with the absence of a no-fly option ($N_m = 0$) and infinite seating on each flight:

Proposition 1: The second-stage fare game always has unique pure strategy Nash equilibrium.

Proposition 2: In the first stage of a two-stage frequency-fare game with constant seats-per-flight and no connecting passengers, assuming the absence of a no-fly option, infinite seating on each flight and two carriers in a single market, each airline's payoff π_i for $i \in \{1, 2\}$ is concave in airline i's frequency strategy, across plausible parameter ranges.[1] That is,

$$\frac{\partial^2 \pi_i}{\partial {f_i}^2} < 0 \text{ for } i \in \{1, 2\}$$

Proposition 3: The first stage of a two-stage frequency-fare game with constant seats-per-flight and no connecting passengers, assuming the absence of a no-fly option, infinite seating on each flight and two carriers in a single market, profit functions are submodular functions in the overall strategy space. That is,

$$\frac{\partial^2 \pi_i}{\partial f_1 \partial f_2} < 0 \text{ for } i \in \{1, 2\}$$

Corollary 1: By changing the sign of the frequency strategy space of one of the carriers, we can trivially convert the function into a *super-modular* function in the overall strategy space. That is,

[1] For the s-curve model, this holds for all cases where ($\alpha < 2.4456$), a conservative bound with respect to empirically estimated values (typically 1.3-1.7), see Belobaba (2016). For the schedule delay model, concavity guarantees depend on both φ and r, see proof of theorem 1 in Appendix I for a discussion of these values.

$$\frac{\partial^2 \pi_i}{\partial f_1 \partial f_2} > 0 \text{ for } i \in \{1, 2\}$$

These results are significant because they demonstrate that subgame-perfect pure strategy Nash equilibrium is a credible and tractable solution concept for our simplified two-stage game. In particular, the existence and uniqueness results indicate the suitability of pure strategy Nash equilibrium as a solution concept for the second-stage game. Taking untransformed frequency strategies to be in the range [ϵ, F], where ϵ is some small positive number less than 1, and F is some large positive number,[2] the supermodularity of the payoff functions puts the first stage game in the class of *supermodular* games. For these games, the existence of a pure-strategy Nash equilibrium is guaranteed, and a broad class of adaptive learning dynamics (including best response dynamics and fictitious play) converge to the set of interval bounded by the largest and smallest Nash Equilibria, as ordered by strategy profile (Milgrom and Roberts, 1990; Chen and Gazzale, 2004). If there is a unique equilibrium, these dynamics converge to it. Taking the same (untransformed) frequency stagy ranges, the game can also be cast as a submodular game. In this case, convergence of simultaneous best response dynamics from the infimum or supremum of the strategy space implies a unique Nash equilibrium and the convergence of the same class of learning dynamics above to it. In the two-player case, since the game can also be cast as a supermodular game, the reverse also holds: uniqueness of the Nash Equilibrium implies the convergence of these learning dynamics. Concave first-stage payoffs, not guaranteed for one-stage models (Hansen 1990), also ensure the existence of a first stage purse-strategy Nash equilibrium independent of supermodularity (Rosen 1965), and additionally ensure that first stage payoff maximization problems (as part of a sequential best response dynamics, for instance) are efficiently solvable and have a unique optimum.

[2] These assumptions are made order to guarantee compactness of strategy spaces and continuity of profit functions. They can be justified by the notion that an airline must have some presence in a market to be considered a player in that market, and that airlines have a finite (though possibly large) supply of craft they can deploy.

This analysis implies that a two-stage approach to modelling frequency and fare competition induces properties in the payoff functions that improve the credibility and tractability of subgame perfect Nash equilibrium. In other words, a more realistic approach to the sequential nature of airline decision making makes a game-theoretic approach to analyzing airline decisions more attractive, both computationally and behaviorally. The existence of these properties in this simple case suggests that more complex models may show some similar favorable properties. Analytical approaches become substantially more difficult as the strong assumptions of this 2-player model are relaxed, so we turn to computational approaches to extend our results to more realistic models.

Numerical Results for Extended Models

We now relax the assumptions of 2 players, absence of a no-fly option and connecting passengers, and unlimited seating, and numerically test the concavity and submodularity results proven above for the simplified case for a range of parameter values. We first compute approximate equilibrium fare vectors for the second stage game for a range of plausible first stage (frequency) strategy profiles. These profiles are generated by combinations of integer daily frequency strategies ranging from 1 flight per day to 20 flights per day. Second stage equilibria were computed by initializing fares for all players at 100, and numerically optimizing each player's payoff (from equation (3)) in turn with respect to fare price. This iterative best rest response heuristic was repeated until fares for each player converged to within a threshold (change from the previous iteration of less than 0.1). This was done for both market share functions (s-curve and schedule delay); for 1, 2 and 3 players; for varying numbers of seats per flight; for connecting passengers; for varying values of the exponential of the utility of the no-fly option; and for varying values of the utility parameters for frequency and fare (β, α for the s-curve model, φ and *r* for the schedule-delay model). Market size *M* was set at 1000 and cost per flight set at 10000. For computational feasibility, parameters were varied one at a time for each number of players, with defaults of α=1.29, β=0.0045, N = 0.5 and number of seats = 125 for the s-curve model, and $r = 0.456$ (as per Douglas and

Miller, 1974), $\varphi = 5.1$ (as per Hansen and Liu 2016), β=0.012 (as per Hansen and Liu 2016), N = 0.005, and numbers of seats = 125 for the schedule delay model. In all cases, this myopic best response heuristic converged to an equilibrium, suggesting that second stage fare equilibria for our model exist in practice across a broad range of scenarios. These results link the analytically demonstrated existence and uniqueness results of Proposition 1 with a broader landscape of more realistic but analytically intractable scenarios. Ranges of varied parameters were chosen to encompass values found in literature and in practice. Tables **2** and **3** list the ranges tested for each parameter and the increments these parameters were varied by for the s-curve and schedule-delay models respectively.

Parameter	**Range Tested**	**Testing Increments**
N	0 to 1	0.1
α	1 to 2	0.1
β	0.001 to 0.01	0.001
Seats-per-flight (S)	25-250, and unlimited seating	25

Table 2: Parameter ranges tested against parameter defaults using s-curve model of market share

Parameter	**Range Tested**	**Testing Increments**
N	0.0001 to 0.005	0.005
r	0.1 to 1	0.1
φ	1 to 10	1
β	0.001 to 0.025	0.001
Seats-per-flight (S)	25-250, and unlimited seating	25

Table 3: Parameter ranges tested against parameter defaults using schedule-delay model of market share

For example, **Figure 2** shows player 1's second stage equilibrium fare values in a 2-player game following the s-curve market share formulation α at 1.29, β at 0.0045, N at 0.5, and unlimited seating.

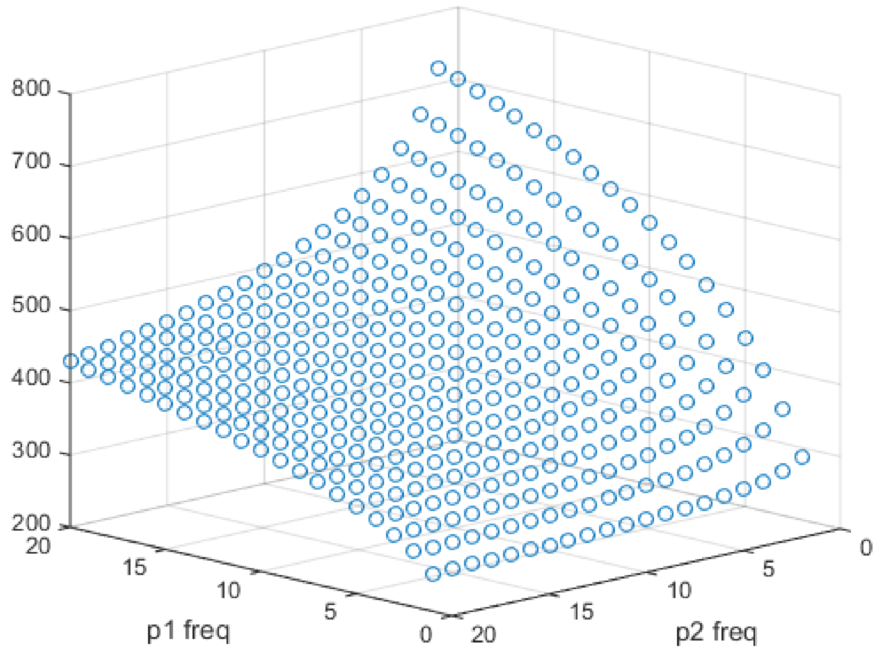


Figure 2: Player 1 fare equilibria for various frequency strategy profiles, with s-curve market share, α=1.29, β=0.0045, N=0.5 and unlimited seating

For each frequency strategy profile and parameter combination, we recorded equilibrium payoffs π_i for $i \epsilon A$. For parameter combination tested and for each player, this generated 20 payoff data points for a monopolistic market, 400 data points for a two-player market, and 8000 data points for a three-player market. We then fit quadratic functions of the market strategy profiles to their respective payoffs using linear regression, for each of these data sets, for each varied parameter combination. Polynomial coefficients of payoff functions were estimated for the following functional forms.

For a monopolistic market, the profit of carrier 1 π_1 flying a daily frequency of f_1 was modeled:

$$\pi_1 \sim \gamma_0 + \gamma_1 f_1 + \gamma_2 {f_1}^2$$

For two-player markets, the model was:

$$\pi_1 \sim \gamma_0 + \gamma_1 f_1 + \gamma_2 f_2 + \gamma_3 {f_1}^2 + \gamma_4 {f_2}^2 + \gamma_5 f_1 f_2$$

For three-player markets, the model was:

$$\pi_1 \sim \gamma_0 + \gamma_1 f_1 + \gamma_2 f_2 + \gamma_3 f_3 + \gamma_4 {f_1}^2 + \gamma_5 {f_2}^2 + \gamma_6 {f_3}^2 + \gamma_7 f_1 f_2 + \gamma_8 f_1 f_3 + \gamma_9 f_2 f_3$$

For a 2-player non-stop and one-stop game: $\pi_1 \sim \gamma_0 + \gamma_1 f_{11} + \gamma_2 f_{12} + \gamma_3 f_{21} + \gamma_4 f_{22} + \gamma_5 {f_{11}}^2 + \gamma_6 {f_{12}}^2 + \gamma_7 {f_{21}}^2 + \gamma_8 {f_{22}}^2 + \gamma_9 f_{11} f_{21} + \gamma_{10} f_{21} f_{22}$, with f_{12}, for example, representing the frequency of player one on the second leg of a one stop market.

Table 4 gives an illustrative example of regression results for a 2-player non-stop s-curve model. In this case, utility parameters are held at defaults (α=1.29, β=0.0045, N=0.5) and the number of seats per flight are varied to encompass large and small craft.

$Seats$	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	R^2
250	122200.0	18135.0	-17856.0	-494.4	686.1	-533.0	0.955
225	122250.0	18130.0	-17861.0	-494.2	686.2	-532.9	0.955
200	122400.0	18115.0	-17876.0	-493.9	686.6	-532.4	0.955
175	122640.0	18095.0	-17901.0	-493.4	687.2	-531.6	0.955
150	123470.0	18030.0	-17989.0	-492.1	689.6	-529.0	0.955
125	125340.0	17925.0	-18214.0	-490.9	696.0	-523.3	0.950
100	129430.0	17885.0	-18838.0	-495.5	716.3	-514.1	0.939
75	136710.0	18277.0	-20301.0	-512.8	773.0	-518.1	0.928
50	142620.0	20224.0	-22355.0	-567.2	865.2	-578.9	0.927
25	104880.0	27814.0	-18929.0	-743.5	773.0	-846.6	0.969

Table 4: Regression coefficients and model R^2 for 2-player s-curve-model, non-stop passengers case with α=1.29, β=0.0045, N=0.5, and different seats-per-flight values.

The coefficient of determination R^2 for fitted models remained > 0.9 for nearly all parameter combinations tested, ranging from close to 0.9 on the low end (in 3 player games with 25 seats per flight, an extreme parameter value) to very close to 1. Exceptions were found in extreme or nonsensical parameter combinations: for example, in 1 player s-curve-model markets with very high β (0.009 or 0.01) R^2 dipped to 0.88 and 0.87 respectively, while in 1-player and 3-player s-curve markets with N=0 (absent no-fly option), R^2 fell to 0.89 and 0.08 respectively. The uniquely poor fit found in the monopolistic markets with no no-fly option is not surprising, as in such markets a carrier unrealistically has no incentive to not charge unreasonably high ticket prices. The generally high R^2 values found suggest that in nearly all cases, a quadratic function of carrier frequencies is able to capture a significant portion of the variation in equilibrium profits, and can provide a reasonable numerical approximation of the payoff functions described in equation (3), both for simple and more realistic cases. This gives us a tool to probe the robustness of the concavity and sub-modularity properties described in Propositions 2 and 3. Examining approximated payoff functions, we find that in all models with high R^2 (>0.9), the signs of estimated coefficients are consistent with both submodularity and concavity. For example, for the two player case, γ_3, the coefficient of the square of player 1's daily frequency, and γ_5, the coefficient of the interaction term $f_1 f_2$, are negative, consistent with concavity and submodularity respectively. Note that this is the case across the range of seat values in **Table 4. Figure 3** shows a quadratic approximation surface fitted to second-stage equilibrium payoffs (represented by points) with an R^2 of 0.95: note that the approximation captures the concavity of player 1's profit with respect to its frequency strategy. For a more extensive enumeration of coefficient estimates and R^2 values for varying parameter values, see Appendix II. While longer computational times precluded extensive parameter sensitivity tests for more than 3 players, more limited testing of 4 player games revealed similar results: good quadratic function approximations and coefficient estimates consistent with concavity and sub modularity. We also examined higher order polynomial approximations for even closer fits to payoff functions: quartic approximations tested on several models retained submodularity and concavity properties. For the remainder of this paper, however, we will focus on quadratic approximations, as these allow for generally good approximations while remaining convenient for simple evaluation of function properties and keeping the number of parameters in check when calibrating models with real-word data.

The robustness of sub-modularity and concavity properties in approximated payoff functions across a wide range of scenarios and parameter values extends the analytical results of Proposition 1 and 2 to a much richer and more realistic class of models. These results suggest that in general, sub-game perfect Nash equilibrium remains a highly tractable and credible solution concept for our game. That property of submodularity extends to more complex scenarios is consistent with the observation that games with this property tend to arise in strategic situations where there is competition for a resource (Roy and Sabarwal, 2012): in this case, that resource is market share. While we cannot extend supermodularity trivially beyond the two-player case (as we no longer simply change the sign of one player's strategy space to flip the sign of the cross derivative consistently), analogy with the two-player case, as well as a growing body of literature on games of strategic substitutes (e.g. Jensen, 2010; Roy and Sabarwal, 2012) provide us with a baseline for further exploration of the convergence of learning dynamics. Concave payoffs maintain the guarantee of the existence of first stage frequency equilibrium, and our quadratic approximations provide a simple mechanism for checking the uniqueness of first stage equilibrium using Rosen's diagonal strict concavity condition. We find that with a few exceptions in extreme parameter values (high $\alpha, > 1.7$), estimated coefficients across parameter ranges tested are consistent with a guaranteed unique first-stage equilibrium. Furthermore, concavity means that individual player bet responses have a unique and easily solved-for optimum, meaning that practical solution heuristics such as myopic best response (a method relying on individual payoff optimizations, employed in previous approaches to airline frequency competition, see for example Vaze and Barnhart, 2012a; 2012b) can be deployed efficiently to find equilibria, even in large scenarios and networks. In the two player case, we can use supermodularity and the uniqueness of first stage Nash equilibria to guarantee the rapid convergence of a broad class of adaptive dynamics to Nash equilibria in the first stage game (following Milgrom and Roberts, 1990) with our approximated payoff functions. We can leverage concavity, submodularity, the polynomial nature of the approximated payoffs, and results from Jensen (2010) to demonstrate the convergence of myopic best response for larger numbers of players:

Proposition 4: An N-player game with quadratic, concave, and submodular payoff functions, belongs to the class of generalized quasi-aggregative games (as defined by Jensen, 2010), and the *myopic best response* search heuristic converges to Nash Equilibrium.

Proof Let the payoff function of player $i \in K$, for strategy profile **s,** be

$$\pi_i(s) = \gamma_0 + \sum_{k \in K} \gamma_k s_k + \sum_{k \in K} \beta_k s_k^2 + \sum_{\substack{j,k \in K \\ j \neq k}} \alpha_{j,k} s_j s_k \quad (4)$$

Let π_i be concave and submodular, such that $\beta_i < 0$ and $a_{ij} < 0 \; \forall \, j \neq i \in K$

Taking $g(\boldsymbol{s}) = \pi_i(\boldsymbol{s})$,

$$\frac{\partial g(\boldsymbol{s})}{\partial s_i} = \gamma_i + 2\beta_i s_i + \sum_{k \in K \neq i} \alpha_{i,k} s_k \quad (5)$$

$\frac{\partial g(s)}{\partial s_i}$ can be written as $f_i(\sigma_i(s_{-i}), s_i)$, where $\sigma_i(s_{-i}) = \sum_{k \in K \neq i} \alpha_{i,k} s_k$ and $f_i(\boldsymbol{s}) = \gamma_i + 2\beta_i s_i + \sigma_i(s_{-i})$, so this game is a generalized quasi-aggregative game as defined by Jensen (2010). As the game is strictly submodular, and $f_i(\sigma_i(s_{-i}), s_i)$ is strictly decreasing in s_{-i}, the game is a best reply potential game (Jensen 2010, Theorem 1), and since best reply functions are single-valued (payoffs being strictly concave), iterated best-reply dynamics converge to the set of PSNEs (Jensen 2010, Theorem 2).

In fact, it can be seen that higher order polynomial payoffs with submodular and concave properties fit into this a generalized quasi-aggregative framework, provided that interaction terms do not contain higher-order products of s_i (${s_i}^2$ for example). This flexible framework may provide future avenues for using polynomial payoff approximation function of this type for otherwise analytically intractable games.

Further results in the theory of games with strategic substitutes by Roy and Sabarwal (2012) demonstrate that the convergence of *simultaneous* best-reply dynamics in such games from the infimum or supremum of the strategy set implies a unique equilibrium and the convergence of a broad class of adaptive dynamics defined by Milgrom and Roberts (1990) to that equilibrium. While the convergence pf *sequential* best reply dynamics (as demonstrated in Proposition 4) do not necessarily imply this, we can explicitly test for convergence such scenarios computationally using our fitted quadratic functions across a range of parameters. In numerical tests of *simultaneous* best response convergence, 2-player quadratic payoff games converge, as one would expect from their supermodularity. 3-player games sometimes reach cycles, however, suggesting that while *sequential* best response dynamics will converge for these games, other learning dynamics may not.

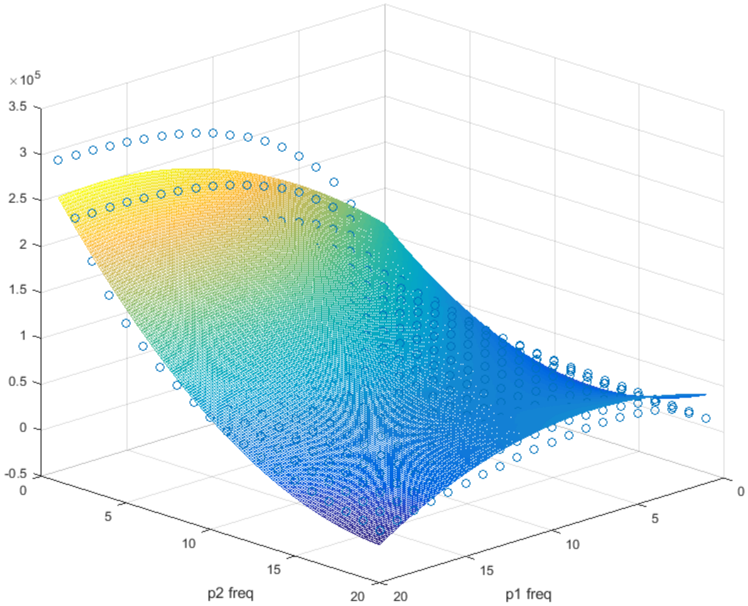


Figure 3: Fit player 1 profit surface for player 1 vs. player 2's frequency strategies, α=1.29, β=0.0045, N=0.5, and unlimited seating. R^2 of model is 0.95

The convergence of learning dynamics in games played with our approximated payoff functions is reassuring both from an intuitive and a computational perspective. The introduction of a second stage fare game has given our payoff functions properties consistent with the convergence of backward-learning dynamics, and allows us to think of our airlines making decisions in less-than perfectly rational terms. In this since, our model incorporates both a forward looking aspect (approximate consideration of future fare decisions) and a backward looking aspect (learning dynamics) in the competitive capacity allocation decisions of airlines. From a computational point of view, fast convergence of easily implementable learning algorithms and easily solvable iterative payoff maximizations enables efficient solutions, experimentation, and calibration of our model when comparing its decision predictions to observed behavior. In the next section, we leverage these properties to apply our model to a real-world airline network.

Airline Network Case Study

To test the tractability and predictive validity of our model in practice, we apply our game-theoretic model to a network of airports in the western United States (SEA, PDX, SFO, SAN, LAX, LAS, PHX, OAK, ONT, SMF, and SJC). We estimate the daily non-stop flight frequencies of the four major carriers in the network (Alaska Airlines - AS, United Airlines - UA, US Airways - US, Southwest Airlines - WN) in the markets in which they are present by computing Nash equilibrium using concave, submodular quadratic functions to approximate the payoffs described by equation (3). See **Figure 4** for an illustration of the networks of each carrier under consideration.

Quadratic payoff functions are constructed for each valid carrier-market combination depending on the number of carriers in the market, based on actual cost and market size data taken from the Bureau of Transportation Statistics (BTS) records from the first quarter of 2007. In particular, payoff function approximations computed for default parameters and market sizes above were transformed by the costs and demands in each particular market, a simple transformation given the functional form of equation (3) with respect to cost and market size. Costs and air operating hours for different craft for different carriers was taken from the Schedule P-5.2 tables (BTS, 2016c). Data on market size, observed frequencies and flight distances was taken from the T100 Segments tables (BTS, 2016a). Data from unidirectional markets containing the same airports were averaged, such that, for example PDX-SAN and SAN-PDX were treated identically for payoff function generation and frequency estimation purposes, as passenger flows, observed frequencies, and other data were generally quite close between differently ordered airport pairs. For simplicity, carrier-market combinations with a carrier market-share of less than 10% or average daily frequency of less than 0.5 were removed from consideration. Finally, three markets, PDX-SJC, PDX-SFO, OAK-PDX have been temporarily removed from consideration, as they contain significant daily frequencies from both Alaska Airlines (AS) and its regional sister carrier Horizon Air (QX). Taking into account such alliances in game-theoretic frequency estimation will require additional modelling considerations. In order to account for connecting passengers, market demands for each carrier-segment combination were adjusted by the ratio of non-stop passengers in the market + connecting passengers for that carrier segment to the total number of passengers flown in that market, such that carriers do not compete for the connecting passengers of other carriers. Data for connecting passenger to total passenger ratios were taken from DB1B Markets and DB1B Coupons tables (BTS, 2016b).

Figure 4: Origin-destination markets served by each of the four airlines (UA, AS, US, WN) considered in this case study

We now use the myopic best response heuristic, justified analytically and numerically above, to solve the game model, with each airline iteratively optimizing a vector of frequencies in all its markets by summing the payoff functions from each of these markets until estimated frequencies converged within a tolerance threshold. Frequency choices were constrained by the estimated availability of various aircraft types to the airline within the network. A single aircraft type was assumed for each carrier-segment combination. Where appropriate, multiple aircraft types were merged into synthetic aircraft type for the purposes of computing constraints, such that each aircraft type appeared in only one synthetic aircraft type, and such that operating costs in each market reflected the proportional composition of aircraft used in that market. To estimate aircraft availability, it was assumed that carriers generally utilize aircraft close to the limits of availability. Thus, the number of aircraft in the network of a type *k* available to a certain airline *a* was calculated as

$$F_{k,a} = \frac{\sum_{m \in M} 2 f_{k,a}(b_m + t)}{18}$$

Where *M* the set of markets where airline a uses craft type *k*, *f* is the observed frequency of *a* on *m*, *t* is the turnaround time of the aircraft (taken to be 30 minutes in all cases), b_m is the average block hours of airline *a* flying market *m*, the factor of 2 accounts for the combined directional markets, and the divisor of 18 accounts for 18 hours of available flying time in the day. During iterative optimization, these fleet size restrictions were applied such that

$$\sum_{m \in M} 2(b_m + t)\hat{f}_{k,a,m} \leq 18F_{k,a}$$

Where $\hat{f}_{k,a}$ is the frequency of being estimated by the model for market *m* for airline *a*. During the myopic best response, regional carriers and American Airline's single frequency on the network were left fixed in order to control for their possibly idiosyncratic behavior, but these fixed frequencies never-the-less contributed to the strategic considerations of other carriers. In all, this left 68 carrier-market combinations to estimate frequencies for. As the myopic best response algorithm is run, players allocate flight frequencies across their respective networks by solving a constrained quadratic program each best response iteration, continuing until convergence. Frequency decision vectors for each player are initialized at 0. The model typically converges in 6-7 iterations (each consisting of four best response optimizations), and is solved in less than one second using MATLAB quadratic programming functions. **Figure 5** plots the convergence of the algorithm for two different convergence thresholds.

In order to calibrate our model, we adjust payoff coefficients to minimize the Mean Absolute Percentage Error, or MAPE, between estimated and observed frequencies over the whole system. MAPE is calculated as

$$MAPE = \frac{\sum_{cm \in CM} \left| \hat{f}_{cm} - f_{cm} \right|}{\sum_{cm \in CM} f_{cm}}$$

Here *CM* is the set of all market-carrier combinations, $\hat{f}$ is the estimated frequency for the market-carrier combination, and f is the observed frequency of the market-carrier combination. For the purposes of calibration, segment carrier combinations were sorted into four groups: three-player markets, two-player markets where both airports were hubs for the carrier, other two-player markets, and monopolistic markets. The resulting 11 coefficients of the payoff functions of these groups (the linear, quadratic, and interaction term coefficients of frequency for the carrier in question, 3 each for 3-player markets and the two 2-player markets, and 2 for monopolistic markets lacking an interaction term), are adjusted simultaneously before transformation by cost and market size data for each carrier-segment combination. These coefficients were adjusted using a gradient approximation algorithm called SPSA (Simultaneous Perturbation Stochastic Approximation, from Spall, 1998) to minimize overall MAPE. Specifically, during each iteration of SPSA, a single game was solved, with payoff coefficients perturbed according to an approximated gradient with respect to the MAPE loss function. SPSA was chosen for its ability to approximate gradient using only two measurements of the loss function (a measurement of MAPE for one convergence of myopic best response), independent of the number of variables being optimized. The 11 coefficients were initialized using values estimated when fitting quadratic functions of frequency to payoff discussed above, using the s-curve market share model with the following parameter values: α=1.29, β=0.0045, N=0.5 and unlimited seating. The game is then run repeatedly until approximate convergence of MAPE, over the course of roughly 10,000 iterations. The best performing coefficients are then used to estimate frequencies across the network, from which we can evaluate in-sample and out-of-sample model performance.

Figure 6 compares actual frequencies (x-axis) and these predicted frequencies (y-axis) in the left panel. The 45º blue line represents equivalent observed and predicted frequencies; most segment predictions are near this line. An overall in-sample MAPE of 18.4% is achieved: more concretely, this corresponds to 49% of absolute prediction errors being less than 1, and 78% being less than 2. Notable outliers were the three highest frequency segments, all hub-to-hub airport segments flown by Southwest Airlines (circled in **Figure 6**). Fixing these frequencies, removing them *MAPE* computation and re-solving the game, we achieve an *MAPE* of 16.7% (**Figure 6,** right panel), with the corrected misallocations improving predictions throughout the model. This suggests that, except for the under-predictions of the highest frequencies, the model empirically performs well for most carrier-segments. These frequency prediction accuracies can be compared to benchmarks for a single airport, ranging from 14%-20%.

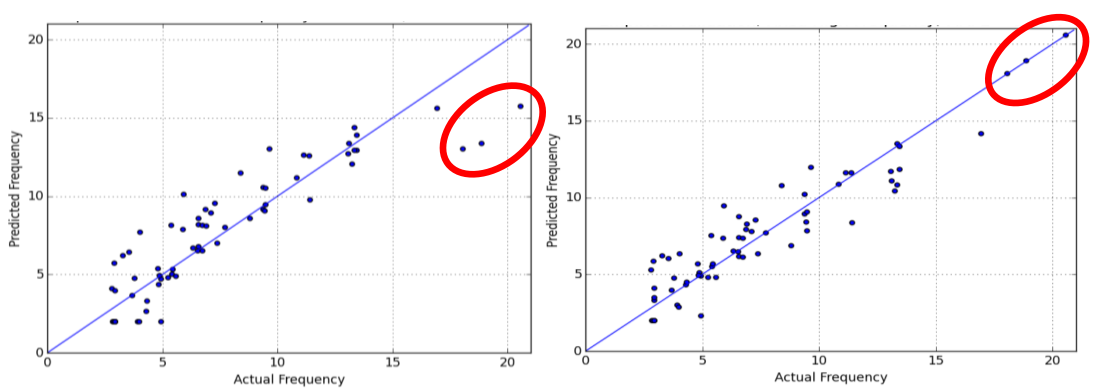


Figure 6: Actual vs. model predicted frequency, with unfixed (left) and fixed (right) high freq. segments,

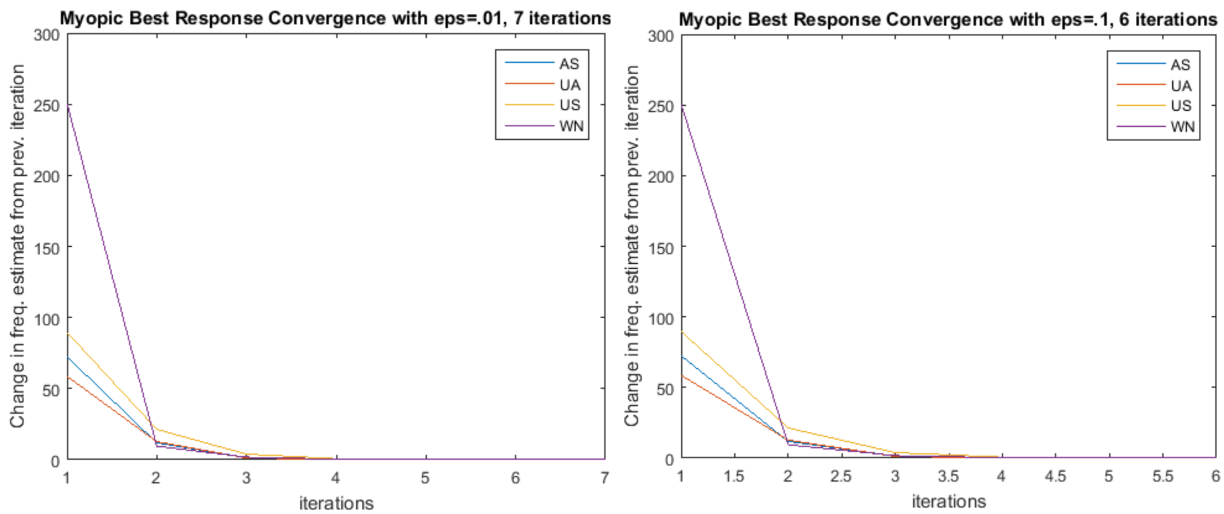


Figure 5 displays the convergence of frequency estimates to equilibrium for each of the four airlines under consideration using the myopic best response heuristic, with two different convergence thresholds (the sum of all differences between estimated frequencies in current and previous iterations).

We can use these same trained coefficients to make frequency predictions out of sample. For example, coefficients trained using SPSA on data from Q1 of 2007 can be used to predict frequencies for Q4 of 2007. As before, we remove from the testing data set markets with allied players and markets with more than 3 players (these did not exist in Q1 after market share cut-offs are made, but do exist in Q4), so that the same restricted set of parameters can be used to mitigate overfitting. Running the game, we find an out of sample testing *MAPE* of 20.6%, corresponding to 47% of absolute frequency errors being less than 1, and 73% being less than 2. However, we can leverage our knowledge of training errors when making out of sample predictions. By adjusting our testing predictions for a given carrier-segment by the error for that carrier-segment in the calibration dataset (simply performing no adjustment to carrier segments not extant in the calibration set), we can substantially reduce our out-of-sample *MAPE: w*e find that the *MAPE* in our Q4 2007 predictions falls to just 11.2%.

A more concrete illustration of this model's out-of-sample prediction accuracy can be found by looking at a new market that arises between Q1 and Q4 of 2007. PDX-SFO is not seen in the calibration data (due to the presence of both QX and AS), yet in Q4 is a duopoly market shared by AS and UA. The frequencies predicted for this market are a good approximation for observed behavior, with an overall *MAPE* of 16.5%, as seen in **Table 5**:

PDX-SFO, Q4 2007 Coefficients calibrated on Q1 2007	**Observed Frequency**	**Predicted Frequency**	**Absolute Error**
UA	6.11	7.34	1.22
AS	3.02	2.74	0.28

Table 5: Out of sample performance on a new market

We can take a another view of out-of-sample prediction accuracy by looking at more aggregate measures of prediction performance, at the carrier, coefficient category (1 player, 2-player hub-hub, other 2-player, 3-player), market and airport levels. In our Q4 2007 predictions based on Q1 calibration data, we find excellent predictions at all of these levels, both unadjusted and adjusted according to training error as described above. With respect to total frequencies allocated by each carrier, we find an MAPE of 2.0%, or 1.5% adjusted (corresponding to average absolute errors of 2.71 and 2.11 flights respectively). With respect to total frequencies allocated within each coefficient category, we find an MAPE of 3.0%, or 2.5% adjusted (corresponding to average absolute errors of 4.2 and 3.42 flights respectively). Across the 41 markets for which estimates were made in the network, we find an MAPE of 14.4%, or 6.3% adjusted (corresponding to average absolute errors of 1.95 and 0.86 flights respectively) for the number of daily flights in each. Across the 11 airports in the network, we find an MAPE for total flights passing through each of 7.8%, or 2.6% adjusted (corresponding to average absolute errors of 7.75 and 2.59 flights respectively). **Table 6** displays the (rounded) error adjusted predictions and actual airport total daily flights for Q4 2007. Predictions at each of these levels of aggregation may be of interested for airlines, airports and other policy makers.

Airport	Observed Flights	Predicted Flights
LAX	128	132
SJC	64	61
LAS	167	168
SAN	108	110
SMF	71	70
SEA	105	104
PDX	41	43
SFO	92	99
ONT	61	62
PHX	159	153
OAK	88	85

Table 6: Out of sample predictions of daily flights deployed at all airports in network, Q4 2007. Calibrated on Q1 2007 data.

In order to examine the predictive accuracy of our model more broadly, we can examine in-sample and out of-sample prediction across years and for varying degrees of look ahead in prediction. In order to do this, we calibrate our 11 coefficients on every quarter from 2007 to 2014, giving us 32 sets of coefficients, and using these coefficients to predict frequencies at every quarter after each of these calibration dates. In this expanded set of data, we include new major carriers and new hubs in the network as appropriate for the date in question.

Examining the unadjusted *MAPE* at varies look-ahead values (i.e. number of quarters ahead the prediction is made with respect to the quarter on which the data is calibrated), we find an almost monotonic increase in median error. **Figure 7** shows *MAPE* for each possible calibration-validation combination (red circles), and the median *MAPE* for that look ahead value in blue (with error bars at one standard deviation). Median *MAPE* remains below 25% for 15 quarters out, suggesting reasonable predictive accuracy in the short and medium term. However, by adjusting predictions by calibration errors in the manner described above, we can achieve significant improvements in *MAPE.* Median *MAPE* in this case remains below 15% for several quarters into the future, though a similar overall increase in median *MAPE* is seen (**Figure 8**). We can also look at median *MAPE* and average absolute error for new markets (**Figure 9** and **10,** respectively). i.e. markets not seen in the calibration data. Median *MAPE* remains close to 20% for several quarters in to the future, suggesting that the new market displayed in **Table 5** is not a highly abnormal short term prediction, and providing further support for the short-to-medium term predictive accuracy of the model.

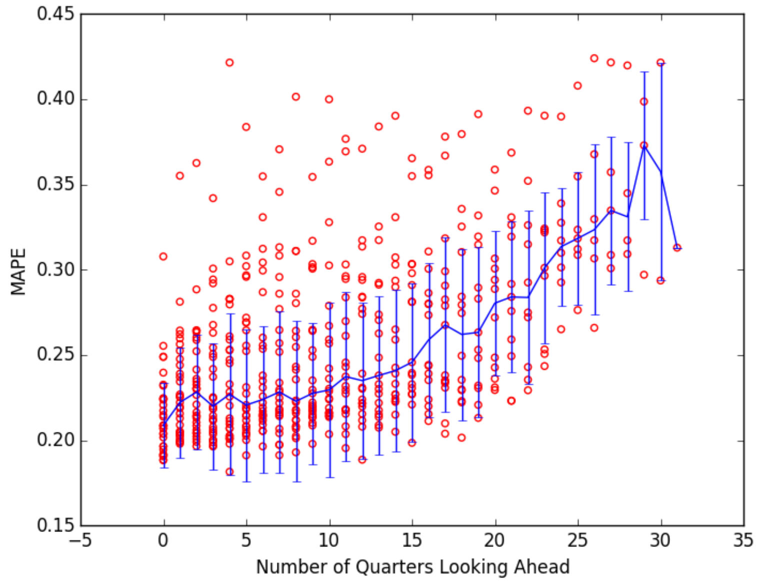


Figure 7: Unadjusted MAPE for varying lookahead

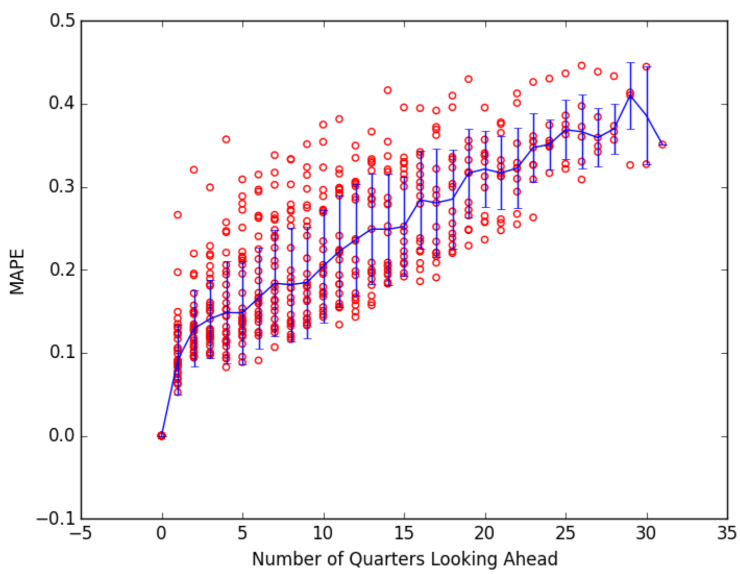


Figure 8: Calibration error-adjusted MAPE for varying lookahead

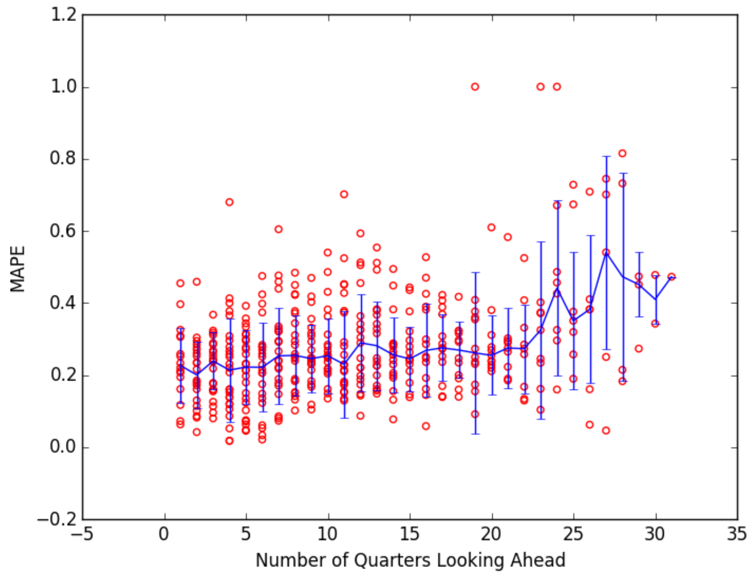


Figure 9: MAPE for new markets for varying lookahead

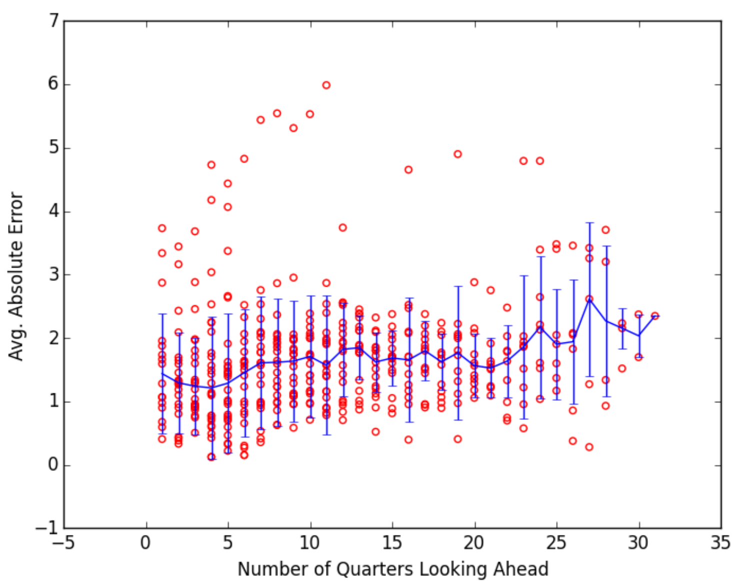


Figure 10: Average absolute error for new markets for varying lookahead

Conclusions

This study investigates a two-stage frequency-fare game-theoretic model of airline competition which is behaviorally consistent with the sequential nature of airline capacity and fare decisions. For simple cases, the analytical qualities of this model indicate well behaved and tractable games, with unique equilibria and convergence properties. Using polynomial payoff function approximations, these properties can be shown numerically to extend to more realistic formulations of the game. In practice, when applied to a real airline network, the model converges quickly and generates daily frequency predictions that closely approximate actual airline decisions, both in sample and out of sample within a short-to-medium term time horizon.

We believe that our model presents multiple avenues for application and future research. To the best of our knowledge, this is the first study to investigate the favorable properties discussed within the context of a two-stage model of airline competition, providing analytical, computational, and empirical results for a game-theoretic approach that has received limited attention in the airline competition literature. We hope that our results presented here can serve as a foundation for a further research into sequential models of airline decision making under competition. Furthermore, the predictive performance of our model on real world data suggests that refinements of the model could serve as a scenario analysis tool to aid in planning, forecasting and policy-making decision support. The tractability of this model and the flexibility with which different scenarios can be tested suggest its potential for rapid and interpretable experimentation in even large-scale airline networks. Here we have considered a relatively simple model of airline competition, without taking into account factors such as market segmentation between business and leisure passengers, passenger loyalty, behavioral differences between airlines and between the non-frequency-fare services they provide, and characteristics of markets beyond cost, observed passenger flow, and hub presence. The fact that our simple model provides a good approximation of airline frequency allocation suggestions that more flexible parameterizations in calibration and prediction could be promising avenues for practitioners.

More broadly, we believe that the general approach presented in this paper could be usefully employed in other domains of applied game theory. Here we have presented a theory-motivated framework for informing a game-theoretic predictive model with data. Game-theoretic models often become intractable for analytical exploration as they are extended to increasingly realistic scenarios. Polynomial approximations of profit functions may provide a convenient method for extending the analytical results of simple game theoretic models to more realistic scenarios, and provide a bridge from theoretical results to models that can be easily calibrated using increasingly available data on strategic behavior. The polynomial approximation approach may provide simple ways to evaluate the properties of otherwise inscrutable models by providing simple windows in properties such as sub/supermodularity and generalizations of aggregative games. We hope that such an approach, within the two-stage framework in particular, could be extended to other applied domains where capacity and pricing decisions are made on different time scales. Earlier work has explored such two-stage capacity and price decisions in models motivated by the telecommunications industry (Acemoglu, Bimpikis, and Ozdaglar, 2006): it is possible that parallels with such domains and the work here could be fruitfully explore in the future.

Appendix I: Full Proofs for s-curve and schedule delay models

Proof of Concavity – s-curve model

Here we give a proof that in a market with 2 competing airlines, all nonstop flights, infinite seating capacity, and the absence of a no-fly option, profit π_i for each airline $i \epsilon \{1, 2\}$ is a concave function of the frequency of airline *i* in that market. With f_i as the frequency and p_i as the fare price of each airline and with α and β as constants, the market share for airline 1 is given by:

$$MS_1 = \frac{e^{\alpha \, ln(f_1) - \beta p_1}}{e^{\alpha \, ln(f_1) - \beta p_1} + e^{\alpha \, ln(f_2) - \beta p_2}}$$

With M as market size, and c as the operating cost of a flight, the profit of airline 1 can then be written as

$$\pi_1 = Mp_1 \left(\frac{e^{\alpha \, ln(f_1) - \beta p_1}}{e^{\alpha \, ln(f_1) - \beta p_1} + e^{\alpha \, ln(f_2) - \beta p_2}} \right) - cf_1 \quad \textbf{(1)}$$

Differentiating, and putting MS_1 in terms of its complementary market share as 1 - MS_2,

$$\frac{\partial \pi_1}{\partial p_1} = M\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1}}{e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2}}\right) - Mp_1 \frac{\partial}{\partial p_1}\left(1 - \frac{e^{\alpha\, ln(f_2) - \beta p_2}}{e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2}}\right)$$

$$\frac{\partial \pi_1}{\partial p_1} = M\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1}}{e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2}}\right) - Mp_1\beta\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1}\, e^{\alpha\, ln(f_2) - \beta p_2}}{(e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2})^2}\right)$$

At $\frac{\partial \pi_1}{\partial p_1} = 0$,

$$M\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1}}{e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2}}\right) = Mp_1\beta\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1}\, e^{\alpha\, ln(f_2) - \beta p_2}}{(e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2})^2}\right)$$

$$\left(\frac{e^{\alpha\, ln(f_2) - \beta p_2}}{e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2}}\right) = \frac{1}{\beta p_1} = MS_2 \quad (\mathbf{2a})$$

Repeating the process for π_2, we also have

$$\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1}}{e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2}}\right) = \frac{1}{\beta p_2} = MS_1 \quad (\mathbf{2b})$$

Additionally, since market shares are complementary,

$$\frac{1}{\beta p_1} + \frac{1}{\beta p_2} = 1 \Rightarrow \frac{1}{p_1} + \frac{1}{p_2} = \beta \Rightarrow p_1 + p_2 = \beta p_1 p_2$$

$$\frac{p_1}{p_2} = \beta p_1 - 1 \quad (\mathbf{3})$$

Plugging $(\mathbf{2b})$ into $(\mathbf{1})$**,** we have

$$\pi_1 = \frac{Mp_1}{\beta p_2} - cf_1$$

Substituting $(\mathbf{3})$ into the above,

$$\pi_1 = \frac{M}{\beta}(\beta p_1 - 1) - cf_1 = Mp_1 - \frac{M}{\beta} - cf_1 \quad \mathbf{(4)}$$

Thus, $sgn\left(\frac{\partial^2 \pi_1}{\partial f_1{}^2}\right) = sgn\left(\frac{\partial^2 \mathrm{p}_1}{\partial f_1{}^2}\right)$ $\mathbf{(5)}$

Dividing $(\mathbf{2b})$ by $(\mathbf{2a})$, we also get

$$\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1}}{e^{\alpha\, ln(f_2) - \beta p_2}}\right) = e^{\alpha(ln(f_1) - ln(f_2)) - \beta(p_1 - p_2)} = \frac{p_1}{p_2}$$

Taking the log of both sides and substituting $\frac{p_1}{p_2} = \beta p_1 - 1$ from $(\mathbf{3})$,

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \beta(p_1 - p_2) + ln(\beta p_1 - 1)$$

Substituting $\frac{p_1}{\beta p_1 - 1} = p_2$ from $(\mathbf{3})$,

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \beta\left(p_1 - \frac{p_1}{\beta p_1 - 1}\right) + ln(\beta p_1 - 1)$$

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1) \quad \mathbf{(6)}$$

Differentiating both sides with respect to f_1,

$$\alpha\left(\frac{f_2}{f_1}\right)\frac{1}{f_2} = \frac{\partial p_1}{\partial f_1}\frac{\partial}{\partial p_1}\left(\frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)\right)$$

$$\frac{\alpha}{f_1} = \left(\frac{\beta^2 p_1}{(\beta p_1 - 1)^2} + \beta\right)\frac{\partial p_1}{\partial f_1}$$

$$\frac{\alpha}{\beta}\frac{1}{f_1} = \left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)\frac{\partial p_1}{\partial f_1} \quad \mathbf{(7a)}$$

$$\frac{\partial p_1}{\partial f_1} = \frac{\frac{\alpha}{\beta}\frac{1}{f_1}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)} \quad (\mathbf{7b})$$

By $(\mathbf{3})$, βp_1 greater than 1, as the price ratio must be positive. In equation $(\mathbf{7a})$ above, the left-hand side and the multiplicands on the right-hand side are positive. Additionally, differentiating the left multiplicand of the right hand side of $(\mathbf{7a})$ with respect to p_1, and with βp_1 greater than 1, we get

$$\frac{\partial}{\partial p_1}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) = \beta \frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4} < 0$$

Differentiating both sides of $(\mathbf{7a})$ with respect to f_1 a second time, we get

$$-\frac{\alpha}{\beta}\frac{1}{{f_1}^2} = \frac{\partial}{\partial f_1}\left[\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)\frac{\partial p_1}{\partial f_1}\right]$$

$$-\frac{\alpha}{\beta}\frac{1}{{f_1}^2} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \beta\left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right)\left(\frac{\partial p_1}{\partial f_1}\right)^2$$

We can rearrange $(\mathbf{7a})$ as $\frac{\partial p_1}{\partial f_1} = \frac{\alpha}{\beta}\frac{1}{f_1}\left(\frac{(\beta p_1 - 1)^2}{\beta^2 {p_1}^2 - \beta p_1 + 1}\right)$, and plugging this expression into the equation above, we have

$$-\frac{\alpha}{\beta {f_1}^2} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \beta\left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right)\frac{\alpha^2}{\beta^2 {f_1}^2}\left(\frac{(\beta p_1 - 1)^4}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^2}\right)$$

$$-\frac{\alpha}{\beta {f_1}^2} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \left(\frac{1 - \beta^2 {p_1}^2}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^2}\right)\frac{\alpha^2}{\beta {f_1}^2}$$

$$-\frac{\alpha}{\beta {f_1}^2} - \left(\frac{1 - \beta^2 {p_1}^2}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^2}\right)\frac{\alpha^2}{\beta {f_1}^2} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)$$

$$\underbrace{\frac{-\frac{\alpha}{\beta {f_1}^2}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)}}_{A}\left[1 + \alpha \underbrace{\left(\frac{1 - \beta^2 {p_1}^2}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^{2}}\right)}_{B}\right] = \frac{\partial^2 p_1}{\partial {f_1}^2} \quad (\mathbf{8})$$

The expression to the left of the square brackets in $(\mathbf{8})$ above (labeled *A*) is negative for all values of $\beta p_1 > 1$, which will be true by $(\mathbf{3})$. Expression *B in* $(\mathbf{8})$ has the form $\left(\frac{1-x^2}{(x^2-x+1)^{2}}\right)$, which has a global minimum of -0.408894 at $x = \beta p_1 = 1.53209$. Thus, for any $\alpha < 2.4456$ (an extreme value for this parameter),

$$\frac{\partial^2 p_1}{\partial {f_1}^2} < 0$$

From $(\mathbf{5})$**,** this implies

$$\frac{\partial^2 \pi_1}{\partial {f_1}^2} < 0$$

The profit function of player 1 is thus concave with respect to the frequency of player 1.

Proof of Unique Pure Strategy Equilibrium in Second Stage – s-curve model

From $(\mathbf{6})$, we have

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)$$

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \beta p_1\left(1 - \frac{1}{\beta p_1 - 1}\right) + ln(\beta p_1 - 1)$$

Taking the right hand side of the equation to be $F(p_1)$ and differentiating with respect to p_1, we have

$$\frac{\partial F(f_1, f_2)}{\partial \beta p_1} = \frac{(\beta p_1)^2 - 2\beta p_1 + 2}{(\beta p_1 - 1)^2} + \frac{1}{\beta p_1 - 1}$$

$\beta p_1 > 1$ by (**3**)**,** and in the above expression,

$$\frac{\partial\, F(p_1)}{\partial \beta p_1} > 0\ \forall\ \beta p_1\ \epsilon\ (1, \infty] \quad (\mathbf{9})$$

β is a positive constant, so is $F(p_1)$ is monotonically increasing on p_1 with $\beta p_1\ \epsilon\ (1, \infty]$.

In addition,

$$\lim_{\beta p_1 \to \infty} F(p_1) = \infty$$

Since $\lim_{\beta p_1 \to \infty} ln(\beta p_1 - 1) \quad = \infty$, and using L'Hopital's rule,

$$\lim_{\beta p_1 \to \infty} \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) = \lim_{\beta p_1 \to \infty} \frac{2\beta p_1 - 2}{1} = \infty$$

Also,

$$\lim_{\beta p_1 \to 1^+} F(p_1) = -\infty$$

Since $\lim_{\beta p_1 \to 1^+} ln(\beta p_1 - 1) \quad = -\infty$, and

$$\lim_{\beta p_1 \to 1^+} \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) = \lim_{\beta p_1 \to 1^+}[(\beta p_1)^2 - 2\beta p_1] \lim_{\beta p_1 \to 1^+} \frac{1}{\beta p_1 - 1} = -\infty$$

Thus, taking the right hand side of (**6**) to be $G(f_1, f_2)$, for any $\{f_1, f_2\}$, there exists an F^{-1} such that $p_1 = F^{-1}(G(f_1, f_2))$ with $\beta p_1 > 1$. By symmetry, the same argument applies for p_2. Thus, for the second stage game, there exists a unique fare vector $({p_1}^*, {p_2}^*)$.

Proof of Submodularity – s-curve model

From $(\boldsymbol{7b})$, we have

$$\frac{\partial p_1}{\partial f_1} = \frac{\frac{\alpha}{\beta}\frac{1}{f_1}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)}$$

Differentiating with respect to f_2, we get

$$\frac{\partial^2 p_1}{\partial f_1 \partial f_2} = \frac{-\frac{\alpha}{\beta}\frac{1}{f_1}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)^2}\frac{\partial}{\partial f_2}\left[\frac{\beta p_1}{(\beta p_1 - 1)^2}\right] = \frac{-\frac{\alpha}{f_1}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)^2}\left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right)\frac{\partial p_1}{\partial f_2}$$

$$\frac{\partial^2 p_1}{\partial f_1 \partial f_2} = \frac{-\frac{\alpha}{f_1}}{(\beta p_1 + (\beta p_1 - 1)^2\,)^2}\left[(1 - \beta^2 {p_1}^2)\frac{\partial p_1}{\partial f_2}\right] \quad (\mathbf{15})$$

On the right-hand side of $(\mathbf{15})$**,** the left multiplicand is < 0 as α and f_1 are both positive. Now we must check the sign of $(1 - \beta^2 {p_1}^2)\frac{\partial p_1}{\partial f_2}$. From $(\mathbf{6})$, we have

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)$$

Differentiating both sides with respect to f_2 ,

$$-\alpha\left(\frac{f_2}{f_1}\right)\frac{f_1}{{f_2}^2} = \frac{\partial p_1}{\partial f_2}\frac{\partial}{\partial p_1}\left(\frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)\right)$$

$$-\frac{\alpha}{f_2} = \beta\frac{\partial p_1}{\partial f_2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) \quad (\mathbf{16})$$

Thus, in $(\mathbf{16})$ above, $\frac{\partial p_1}{\partial f_2} < 0$, so $(1 - \beta^2 {p_1}^2)\frac{\partial p_1}{\partial f_2} < 0$ if $\beta p_1 > 1$. This is the case at equilibrium, so from $(\mathbf{15})$ we have

$$\frac{\partial^2 p_1}{\partial f_1 \partial f_2} < 0 \quad (\mathbf{17})$$

From $(\mathbf{4})$, we have

$$\pi_1 = Mp_1 - \frac{M}{\beta} - cf_1$$

So $(\mathbf{17})$ also implies

$$\frac{\partial^2 \pi_1}{\partial f_1 \partial f_2} < 0 \quad (\mathbf{18})$$

Proof of Concavity – schedule delay model

Here we give a proof that in a market with 2 competing airlines, all nonstop flights, infinite seating capacity, and the absence of a no-fly option, profit π_i for each airline $i \in \{1, 2\}$ is a concave function of the frequency of airline *i* in that market. With f_i as the frequency and p_i as the fare price of each airline and with *m*, φ, and β as positive parameters, the market share for airline 1 is given by:

$$MS_1 = \frac{e^{-\varphi f_1{}^{-m} - \beta p_1}}{e^{-\varphi f_1{}^{-m} - \beta p_1} + e^{-\varphi f_2{}^{-m} - \beta p_2}}$$

With M as market size, and c as the operating cost of a flight, the profit of airline 1 can then be written as

$$\pi_1 = Mp_1 \left(\frac{e^{-\varphi f_1{}^{-m} - \beta p_1}}{e^{-\varphi f_1{}^{-m} - \beta p_1} + e^{-\varphi f_2{}^{-m} - \beta p_2}} \right) - cf_1 \quad (\mathbf{1})$$

Differentiating, and putting MS_1 in terms of its complementary market share as 1 - MS_2,

$$\frac{\partial \pi_1}{\partial p_1} = M\left(\frac{e^{-\varphi f_1{}^{-m} - \beta p_1}}{e^{-\varphi f_1{}^{-m} - \beta p_1} + e^{-\varphi f_2{}^{-m} - \beta p_2}}\right) - Mp_1 \frac{\partial}{\partial p_1}\left(1 - \frac{e^{-\varphi f_2{}^{-m} - \beta p_2}}{e^{-\varphi f_1{}^{-m} - \beta p_1} + e^{-\varphi f_2{}^{-m} - \beta p_2}}\right)$$

$$\frac{\partial \pi_1}{\partial p_1} = M\left(\frac{e^{-\varphi f_1{}^{-m} - \beta p_1}}{e^{-\varphi f_1{}^{-m} - \beta p_1} + e^{-\varphi f_2{}^{-m} - \beta p_2}}\right) - Mp_1\beta\left(\frac{e^{-\varphi f_1{}^{-m} - \beta p_1} e^{-\varphi f_2{}^{-m} - \beta p_2}}{\left(e^{-\varphi f_1{}^{-m} - \beta p_1} + e^{-\varphi f_2{}^{-m} - \beta p_2}\right)^2}\right)$$

At $\frac{\partial \pi_1}{\partial p_1} = 0$,

$$M\left(\frac{e^{-\varphi f_1{}^{-m} - \beta p_1}}{e^{-\varphi f_1{}^{-m} - \beta p_1} + e^{-\varphi f_2{}^{-m} - \beta p_2}}\right) = Mp_1\beta\left(\frac{e^{-\varphi f_1{}^{-m} - \beta p_1} e^{-\varphi f_2{}^{-m} - \beta p_2}}{\left(e^{-\varphi f_1{}^{-m} - \beta p_1} + e^{-\varphi f_2{}^{-m} - \beta p_2}\right)^2}\right)$$

$$\left(\frac{e^{-\varphi f_2{}^{-m} - \beta p_2}}{e^{-\varphi f_1{}^{-m} - \beta p_1} + e^{-\varphi f_2{}^{-m} - \beta p_2}}\right) = \frac{1}{\beta p_1} = MS_2 \quad (\mathbf{2a})$$

Repeating this process for π_2, we also have

$$\left(\frac{e^{-\varphi f_1{}^{-m} - \beta p_1}}{e^{-\varphi f_1{}^{-m} - \beta p_1} + e^{-\varphi f_2{}^{-m} - \beta p_2}}\right) = \frac{1}{\beta p_2} = MS_1 \quad (\mathbf{2b})$$

Additionally, since market shares are complementary,

$$\frac{1}{\beta p_1} + \frac{1}{\beta p_2} = 1 \Rightarrow \frac{1}{p_1} + \frac{1}{p_2} = \beta \Rightarrow p_1 + p_2 = \beta p_1 p_2$$

$$\frac{p_1}{p_2} = \beta p_1 - 1 \quad (\mathbf{3})$$

Plugging $(\mathbf{2b})$ into $(\mathbf{1})$**,** we have

$$\pi_1 = \frac{Mp_1}{\beta p_2} - cf_1$$

Substituting $(\mathbf{3})$ into the above,

$$\pi_1 = M(\beta p_1 - 1) - cf_1 = Mp_1 - \frac{M}{\beta} - cf_1 \quad (\mathbf{4})$$

Thus, $sgn\left(\frac{\partial^2 \pi_1}{\partial {f_1}^2}\right) = sgn\left(\frac{\partial^2 p_1}{\partial {f_1}^2}\right)$ $(\mathbf{5})$

Dividing $(\mathbf{2b})$ by $(\mathbf{2a})$, we also get

$$\left(\frac{e^{-\varphi {f_1}^{-m} - \beta p_1}}{e^{-\varphi {f_2}^{-m} - \beta p_2}}\right) = e^{-\varphi(f_1 - f_2) - \beta(p_1 - p_2)} = \frac{p_1}{p_2}$$

Taking the log of both sides,

$$ln\left(\frac{p_1}{p_2}\right) = -\varphi(f_1 - f_2) - \beta(p_1 - p_2)$$

Substituting $\frac{p_1}{p_2} = \beta p_1 - 1$ from $(\mathbf{3})$,

$$-\varphi({f_1}^{-m} - {f_2}^{-m}) = \beta(p_1 - p_2) + ln(\beta p_1 - 1)$$

Substituting $\frac{p_1}{\beta p_1 - 1} = p_2$,

$$-\varphi({f_1}^{-m} - {f_2}^{-m}) = \beta\left(p_1 - \frac{p_1}{\beta p_1 - 1}\right) + ln(\beta p_1 - 1)$$

$$-\varphi({f_1}^{-m} - {f_2}^{-m}) = \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1) \quad (\mathbf{6})$$

Differentiating both sides with respect to f_1,

$$\varphi m {f_1}^{-(m+1)} = \frac{\partial p_1}{\partial f_1}\frac{\partial}{\partial p_1}\left(\frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)\right)$$

$$\varphi m {f_1}^{-(m+1)} = \left(\frac{\beta^2 p_1}{(\beta p_1 - 1)^2} + \beta\right)\frac{\partial p_1}{\partial f_1}$$

$$\frac{\varphi m {f_1}^{-(m+1)}}{\beta} = \left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)\frac{\partial p_1}{\partial f_1} \quad (\mathbf{7a})$$

$$\frac{\partial p_1}{\partial f_1} = \frac{\frac{\varphi m}{\beta} {f_1}^{-(m+1)}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)} \quad (\mathbf{7b})$$

In equation $(\mathbf{7a})$ above, the left-hand side and the multiplicands on the right-hand side are positive. Additionally, differentiating the left multiplicand of the right hand side of $(\mathbf{7a})$ with respect to p_1, and with βp_1 greater than 1 (which will be the case in the ranges considered), we get

$$\frac{\partial}{\partial p_1}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) = \frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4} < 0$$

Differentiating both sides of $(\mathbf{7a})$ with respect to f_1 a second time, we get

$$\frac{-\varphi m(m+1) {f_1}^{-(m+2)}}{\beta} = \frac{\partial}{\partial f_1}\left[\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)\frac{\partial p_1}{\partial f_1}\right]$$

$$\frac{-\varphi m(m+1) {f_1}^{-(m+2)}}{\beta} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \beta\left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right)\left(\frac{\partial p_1}{\partial f_1}\right)^2$$

We can rearrange $(\mathbf{7a})$ as $\frac{\partial p_1}{\partial f_1} = \frac{\varphi m}{\beta} {f_1}^{-(m+1)}\left(\frac{(\beta p_1 - 1)^2}{\beta^2 {p_1}^2 - \beta p_1 + 1}\right)$, and plugging this expression into the equation above, we have

$$\frac{-\varphi m(m+1) {f_1}^{-(m+2)}}{\beta}$$
$$= \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \beta\left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right)\frac{\varphi^2 m^2}{\beta^2} {f_1}^{-2(m+1)}\left(\frac{(\beta p_1 - 1)^4}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^2}\right)$$

$$\frac{-\varphi m(m+1) {f_1}^{-(m+2)}}{\beta} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \left(\frac{1 - \beta^2 {p_1}^2}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^2}\right)\frac{\varphi^2 m^2}{\beta} {f_1}^{-2(m+1)}$$

$$\frac{-\varphi m(m+1){f_1}^{-(m+2)}}{\beta} - \left(\frac{1-\beta^2{p_1}^2}{(\beta^2{p_1}^2-\beta p_1+1)^2}\right)\frac{\varphi^2 m^2}{\beta}{f_1}^{-2(m+1)} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1-1)^2}+ 1\right)$$

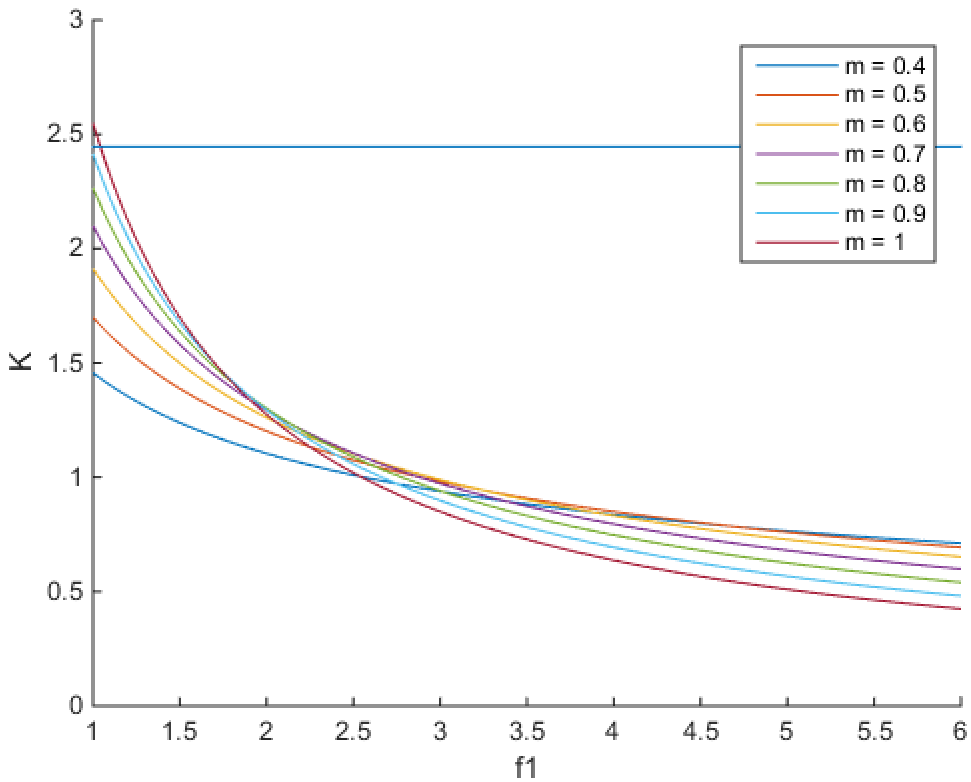
$$\frac{-\varphi m(m+1){f_1}^{-m-2}}{\beta} - \left(\frac{1-\beta^2{p_1}^2}{(\beta^2{p_1}^2-\beta p_1+1)^2}\right)\frac{\varphi^2 m^2}{\beta}{f_1}^{-2m-2} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1-1)^2}+ 1\right)$$

$$\frac{-\varphi m(m+1){f_1}^{-m-2}}{\beta}\left[1+\left(\frac{1-\beta^2{p_1}^2}{(\beta^2{p_1}^2-\beta p_1+1)^2}\right)\frac{\varphi m^2}{m(m+1)}{f_1}^{-2m-2+m+2}\right] = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1-1)^2}+ 1\right)$$

$$\underbrace{\frac{-\varphi m(m+1){f_1}^{-m-2}}{\beta\left(\frac{\beta p_1}{(\beta p_1-1)^2}+ 1\right)}}_{A}\left[1+\frac{\varphi m^2}{m(m+1)}{f_1}^{-m}\underbrace{\left(\frac{1-\beta^2{p_1}^2}{(\beta^2{p_1}^2-\beta p_1+1)^{2}}\right)}_{B}\right] = \frac{\partial^2 p_1}{\partial {f_1}^2}$$

The expression to the left of the square brackets in $(\mathbf{8})$ above (labeled *A*) is negative for all values of $\beta p_1 > 1$, which will be true by $(\mathbf{3})$. Expression *B in* $(\mathbf{8})$ has the form $\left(\frac{1-x^2}{(x^2-x+1)^{2}}\right)$, which has a global minimum of -0.408894 at $x = \beta p_1 = 1.53209$, as in the *s*-curve model. Let $K = \frac{\varphi m^2}{m(m+1)}{f_1}^{-m}$. If K is < 2.4456, $\frac{\partial^2 p_1}{\partial {f_1}^2} < 0$ and from $(\mathbf{5})$**,** this implies $\frac{\partial^2 \pi_1}{\partial {f_1}^2} < 0$, and thus that the profit of player 1 function is concave with respect to the frequency strategy of player 1.

Using $\varphi = 5.1$ estimated by Hansen and Liu (2015), and any value of m less than 0.97, the expression K is < 2.4456 for all values of f_1 1 or greater. m has been cited in literature as 0.456 by Douglas and Miller (1974). Abrahams (1983) takes $m = 1$, in which case concavity holds for all values of $f_1 > 1.043$. Plot below displays K for values of player 1 frequency, for different values of m and $\varphi = 5.1$. The horizontal line represents the value of K below which concavity of player 1's profit function holds.



Proof of Unique Pure Strategy Equilibrium in Second Stage – schedule delay model

From (**6**), we have

$$-\varphi({f_1}^{-m} - {f_2}^{-m}) = \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)$$

$$-\varphi({f_1}^{-m} - {f_2}^{-m}) = \beta p_1 \left(1 - \frac{1}{\beta p_1 - 1}\right) + ln(\beta p_1 - 1)$$

Taking the right hand side of the equation to be $F(p_1)$ and differentiating with respect to p_1, we have

$$\frac{\partial F(f_1, f_2)}{\partial \beta p_1} = \frac{(\beta p_1)^2 - 2\beta p_1 + 2}{(\beta p_1 - 1)^2} + \frac{1}{\beta p_1 - 1}$$

$\beta p_1 > 1$ by $(\mathbf{3})$**,** and in the above expression,

$$\frac{\partial\, F(p_1)}{\partial \beta p_1} > 0 \;\forall\; \beta p_1 \,\epsilon\, (1, \infty] \quad (\mathbf{9})$$

β is a positive constant, so is $F(p_1)$ is monotonically increasing on p_1 with $\beta p_1 \,\epsilon\, (1, \infty]$.

In addition,

$$\lim_{\beta p_1 \to \infty} F(p_1) = \infty$$

Since $\lim_{\beta p_1 \to \infty} ln(\beta p_1 - 1) \quad = \infty$, and using L'Hopital's rule,

$$\lim_{\beta p_1 \to \infty} \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) = \lim_{\beta p_1 \to \infty} \frac{2\beta p_1 - 2}{1} = \infty$$

Also,

$$\lim_{\beta p_1 \to 1^+} F(p_1) = -\infty$$

Since $\lim_{\beta p_1 \to 1^+} ln(\beta p_1 - 1) \quad = -\infty$, and

$$\lim_{\beta p_1 \to 1^+} \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) = \lim_{\beta p_1 \to 1^+} [(\beta p_1)^2 - 2\beta p_1] \lim_{\beta p_1 \to 1^+} \frac{1}{\beta p_1 - 1} = -\infty$$

Thus, taking the right hand side of $(\mathbf{6})$ to be $G(f_1, f_2)$, for any $\{f_1, f_2\}$, there exists an F^{-1} such that $p_1 = F^{-1}(G(f_1, f_2))$ with $\beta p_1 > 1$. By symmetry, the same argument applies for p_2. Thus, for the second stage game, there exists a unique fare vector $({p_1}^*, {p_2}^*)$.

Proof of Submodularity – schedule delay model

From $(\mathbf{7b})$, we have

$$\frac{\partial p_1}{\partial f_1} = \frac{\frac{\varphi m}{\beta} {f_1}^{-(m+1)}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)} \quad (\mathbf{7b})$$

Differentiating with respect to f_2, we get

$$\frac{\partial^2 p_1}{\partial f_1 \partial f_2} = \frac{-\frac{\varphi m}{\beta} {f_1}^{-(m+1)}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)^2} \frac{\partial}{\partial f_2}\left[\frac{\beta p_1}{(\beta p_1 - 1)^2}\right] = \frac{-\varphi m {f_1}^{-(m+1)}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)^2} \left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right) \frac{\partial p_1}{\partial f_2}$$

$$\frac{\partial^2 p_1}{\partial f_1 \partial f_2} = \frac{-\varphi m {f_1}^{-(m+1)}}{(\beta p_1 + (\beta p_1 - 1)^2)^2} \left[(1 - \beta^2 {p_1}^2) \frac{\partial p_1}{\partial f_2}\right] \quad (\mathbf{15})$$

On the right-hand side of $(\mathbf{15})$**,** the left multiplicand is < 0 as φ, m and f_1 are both positive. Now we must check the sign of $(1 - \beta^2 {p_1}^2) \frac{\partial p_1}{\partial f_2}$. From $(\mathbf{6})$, we have

$$-\varphi({f_1}^{-m} - {f_2}^{-m}) = \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)$$

Differentiating both sides with respect to f_2 ,

$$-\varphi m {f_2}^{-(m+1)} = \frac{\partial p_1}{\partial f_2} \frac{\partial}{\partial p_1}\left(\frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)\right)$$

$$-\varphi m {f_2}^{-(m+1)} = \beta \frac{\partial p_1}{\partial f_2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) \quad (\mathbf{16})$$

Thus, in $(\mathbf{16})$ above, $\frac{\partial p_1}{\partial f_2} < 0$, so $(1 - \beta^2 {p_1}^2) \frac{\partial p_1}{\partial f_2} > 0$ if $\beta p_1 > 1$. This is the case at equilibrium, so from $(\mathbf{15})$ we have

$$\frac{\partial^2 p_1}{\partial f_1 \partial f_2} < 0 \quad (\mathbf{17})$$

From $(\mathbf{4})$, we have

$$\pi_1 = Mp_1 - \frac{M}{\beta} - cf_1$$

So $(\mathbf{17})$ also implies

$$\frac{\partial^2 \pi_1}{\partial f_1 \partial f_2} < 0 \quad (\mathbf{18})$$

Thus we have a 2-player submodular game. This can be converted to a supermodular game by converting one airlines strategy from $f_1 \rightarrow -f_1$.

Appendix II: Numerical Demonstrations of Concavity and Sub-modularity for Monopolistic and 2-player Markets, s-curve model

Monopolistic markets

Seats set at 125, α set at 1.29, β set at -0.0045, and N varied

N	β_0	β_1	β_2	R^2
0	1.97601E+20	1.96112E+19	-1.0527E+18	0.0833
0.1	115910.3649	73790.7562	-2474.2687	0.9402
0.2	96026.1366	58831.6758	-1947.2537	0.9467
0.3	83375.1311	50673.6621	-1662.2348	0.952
0.4	73911.7503	45192.1608	-1472.2083	0.9567
0.5	66371.9819	41130.2902	-1332.6892	0.9607
0.6	60078.2569	37944.2811	-1224.2233	0.964
0.7	54657.0087	35349.6833	-1136.6445	0.9671
0.8	49882.4819	33179.4721	-1063.9989	0.9698
0.9	45637.9127	31324.2479	-1002.4305	0.9722
1	41852.1888	29709.2737	-949.322	0.9743

Seats set at 125, β set at -0.0045, N set at 0.5 and α varied

α	β_0	β_1	β_2	R^2
1	76055.8558	27111.8823	-978.6761	0.9184
1.1	73018.5278	31772.9456	-1095.7896	0.9394

1.2	69655.8239	36619.5674	-1218.1896	0.9526
1.3	65992.8556	41639.7952	-1345.6575	0.9614
1.4	62053.1236	46822.2043	-1477.9626	0.9675
1.5	57858.4256	52155.9812	-1614.8682	0.9718
1.6	53419.0015	57631.3392	-1756.1241	0.975
1.7	48708.376	63240.4879	-1901.4358	0.9775
1.8	43740.5398	68974.7085	-2050.5732	0.9795
1.9	38533.8451	74825.7042	-2203.3234	0.9811
2	33105.1127	80785.7979	-2359.4833	0.9824

Seats set at 125, α set at 1.29, N set at 0.5, and β varied

β	β_0	β_1	β_2	R^2
-0.001	428238.2013	198072.3873	-5166.1374	0.9878
-0.002	214119.1007	94036.1936	-2583.0687	0.9849
-0.003	142746.0671	59357.4624	-1722.0458	0.9809
-0.004	107059.5503	42018.0968	-1291.5344	0.9752
-0.005	85647.6403	31614.4775	-1033.2275	0.9669
-0.006	71373.0336	24678.7312	-861.0229	0.9545
-0.007	61176.8859	19724.6268	-738.0196	0.9359
-0.008	53529.7752	16009.0484	-645.7672	0.9096
-0.009	47582.0224	13119.1541	-574.0153	0.8804
-0.01	42823.8201	10807.2387	-516.6137	0.8669

α set at 1.29, β set at -0.0045, N set at 0.5, and with seats-per-flight varied

$Seats$	β_0	β_1	β_2	R^2
250	93652.4801	36515.3194	-1158.9365	0.9698
225	92281.4641	36766.5734	-1168.8284	0.9681
200	89851.7977	37202.3269	-1185.7855	0.966
175	85563.681	37956.7956	-1214.8342	0.9633
150	78455.5194	39169.2144	-1260.6959	0.9609
125	66371.9819	41130.2902	-1332.6892	0.9607
100	45877.5358	44144.4919	-1436.3016	0.9669
75	12697.3648	47810.3588	-1533.0006	0.984
50	-24046.3907	45542.0565	-1280.9701	0.998
25	-6873.6566	18536.7611	-138.0535	0.9998

2-player markets

Seats set at 10000, α set at 1.29, β set at -0.0045, and N varied

N	β_0	β_1	β_2	β_3	β_4	β_5	R^2
0	16470.0	-34936.0	-425.6	1300.0	-595.7	16470.0	0.923

0.1	18555.0	-28020.0	-492.3	1058.2	-605.7	18555.0	0.936
0.2	18942.0	-24161.0	-508.4	918.7	-589.7	18942.0	0.943
0.3	18838.0	-21489.0	-509.1	820.8	-570.3	18838.0	0.948
0.4	18532.0	-19467.0	-503.3	746.0	-551.1	18532.0	0.952
0.5	18135.0	-17856.0	-494.4	686.1	-533.0	18135.0	0.955
0.6	17695.0	-16527.0	-484.0	636.3	-516.2	17695.0	0.957
0.7	17239.0	-15403.0	-473.0	594.1	-500.6	17239.0	0.960
0.8	16779.0	-14436.0	-461.7	557.6	-486.1	16779.0	0.962
0.9	16323.0	-13591.0	-450.4	525.7	-472.7	16323.0	0.964
1	15876.0	-12844.0	-439.3	497.4	-460.1	15876.0	0.965

Seats set at 10000, β set at -0.0045, N set at 0.5 and α varied

α	β_0	β_1	β_2	β_3	β_4	β_5	R^2
0	11435.0	-11441.0	-418.3	419.0	-303.8	11435.0	0.953
0.1	13769.0	-13484.0	-447.6	502.9	-376.5	13769.0	0.953
0.2	16081.0	-15706.0	-473.7	595.5	-456.1	16081.0	0.954
0.3	18361.0	-18104.0	-496.5	696.5	-541.8	18361.0	0.955
0.4	20604.0	-20673.0	-516.0	805.8	-633.1	20604.0	0.955
0.5	22804.0	-23405.0	-532.1	923.0	-729.0	22804.0	0.955
0.6	24960.0	-26292.0	-545.0	1047.6	-828.9	24960.0	0.954
0.7	27073.0	-29320.0	-555.0	1179.0	-932.1	27073.0	0.953
0.8	29147.0	-32478.0	-562.1	1316.8	-1038.2	29147.0	0.952
0.9	31186.0	-35751.0	-566.7	1460.2	-1146.7	31186.0	0.951
1	33195.0	-39125.0	-569.1	1608.8	-1257.4	33195.0	0.950

Seats set at 10000, α set at 1.29, N set at 0.5, and β varied

β	β_0	β_1	β_2	β_3	β_4	β_5	R^2
-0.001	549860.0	116610.0	-80348.0	-2224.9	3087.0	-2398.5	0.975
-0.002	274920.0	53307.0	-40174.0	-1112.5	1543.5	-1199.3	0.969
-0.003	183290.0	32204.0	-26783.0	-741.6	1029.0	-799.5	0.963
-0.004	137470.0	21652.0	-20088.0	-556.2	771.8	-599.6	0.957
-0.005	109980.0	15321.0	-16071.0	-444.9	617.5	-479.7	0.953
-0.006	91640.0	11102.0	-13391.0	-370.8	514.5	-399.8	0.954
-0.007	78548.0	8087.7	-11478.0	-317.8	441.0	-342.7	0.954
-0.008	68730.0	5826.6	-10044.0	-278.1	385.9	-299.9	0.963
-0.009	61095.0	4067.8	-8927.9	-247.2	343.0	-266.5	0.969
-0.01	54988.0	2660.7	-8035.6	-222.5	308.7	-239.9	0.975

α set at 1.29, β set at -0.0045, N set at 0.5, and with seats-per-flight varied

$Seats$	β_0	β_1	β_2	β_3	β_4	β_5	R^2
250	122200.0	18135.0	-17856.0	-494.4	686.1	-533.0	0.955
225	122250.0	18130.0	-17861.0	-494.2	686.2	-532.9	0.955
200	122400.0	18115.0	-17876.0	-493.9	686.6	-532.4	0.955

175	122640.0	18095.0	-17901.0	-493.4	687.2	-531.6	0.955
150	123470.0	18030.0	-17989.0	-492.1	689.6	-529.0	0.955
125	125340.0	17925.0	-18214.0	-490.9	696.0	-523.3	0.950
100	129430.0	17885.0	-18838.0	-495.5	716.3	-514.1	0.939
75	136710.0	18277.0	-20301.0	-512.8	773.0	-518.1	0.928
50	142620.0	20224.0	-22355.0	-567.2	865.2	-578.9	0.927
25	104880.0	27814.0	-18929.0	-743.5	773.0	-846.6	0.969

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